ECONOMIC CONSIDERATIONS ON HOMOGENEOUS LINEAR PRODUCTION FUNCTIONS

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ABSTRACT: *An old problem of economists has been to use linearly homogeneous functions. To obtain notable results, it is desirable that the degree of the respective homogeneous functions be equal at most 1. The advantage of these factors was that an increase in the factors of production by a certain factor leads to a proportional increase in production by the same factor. The paper presents the classic case of the Cobb-Douglas production function with two variables, which are capital and production, and then the paper will focus on two other cases of this type of function with economic applications with corresponding solved examples.*

KEYWORDS: *exponential model, least squares method, Cobb Douglas functions, maximum production at a given budget.*

JEL CLASSIFICATION: *J61.*

1. INTRODUCTION

Definition 1 *We say that a function a function $f: \mathbb{R}^n \to \mathbb{R}$ that checks the property that, for every* $t > 0$

$$f(tx_1, tx_2, \dots, t_n) = t^k f(x_1, x_2, \dots, x_n) \quad (1)$$

is called a homogeneous function of degree k.

Definition 2 *We say that a function $f: \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function it is a homogeneous function of degree I.*

Remark 1 *If we take into account definitions 1 and 2, we deduce that function $f: \mathbb{R}^n \to \mathbb{R}$ is linearly homogeneous if it satisfies the condition that*

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$$f(tx_1, tx_2, \dots, t_n) = tf(x_1, x_2, \dots, x_n), \forall t > 0 \quad (2)$$

Definition 3 *We say that a function* $f: \mathbb{R}^n \to \mathbb{R}$ *is of the Cobb-Douglas type if it has the general form*

$$f(x_1, x_2, \dots, x_n) = \alpha \cdot {x_1}^{\alpha_1} \cdot {x_2}^{\alpha_2} \cdot \dots \cdot {x_n}^{\alpha_n}, \alpha, \alpha_1, \alpha_2, \dots, \alpha_n > 0 \quad (3)$$

Definition 4 *We say that a point* $x^0 \in \mathbb{R}^n$ *is a conditional extreme point (conditional maximum or conditional minimum) for the function* $f: \mathbb{R}^n \to \mathbb{R}$ *with respect to the connection given by the function* $F: \mathbb{R}^n \to \mathbb{R}$ *if is an extreme point (maximum or minimum) for the function f and, in addition, it verifies the additional condition* $F(x^0) = 0$.

Definition 5 *We call the Lagrange function attached to the function* $f: \mathbb{R}^n \to \mathbb{R}$ *which has the connection* $F(x^0) = 0$ *, where* $F: \mathbb{R}^n \to \mathbb{R}$ *, the function* $L: \mathbb{R}^{n+1} \to \mathbb{R}$ *given by the relation*

$$L(x_1, x_2, \dots, x_n, \lambda) = f(x_1, x_2, \dots, x_n) + \lambda F(x_1, x_2, \dots, x_n) \ . \quad (4)$$

Remark 2 *Determining such conditional extremum points of the function f comes down to finding the so-called stationary points of the function L which are the solution of the system*

$$\begin{cases} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \dots \\ \frac{\partial L}{\partial x_n} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \quad (5)$$

2. THEORETICAL ASPECTS

Linearly homogeneous production functions are mathematical models applied in economic analysis to describe the relationship between factors of production and the resulting output and the study of the present paper is a concretization of some studies described in [3].

These functions exhibit what we call *constant returns to scale,* which means that multiplying the factors of production by a constant $\lambda > 0$ will increase output by the same amount $\lambda > 0$.

Regarding the content of this paper we will work with a particular case of Cobb-Douglas type function given by (3) namely those that check the condition

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1 \quad (6)$$

Remark 3 *A Cobb-Douglas type type function given by (3) and which verifies the particular condition given by (4) is a homogeneous function.*

This is easy to verify because

$$f(tx_1, tx_2, \dots, tx_n) = \alpha \cdot (tx_1)^{\alpha_1} \cdot (tx_2)^{\alpha_2} \cdot \dots \cdot (tx_n)^{\alpha_n} =$$

$$= t^{\alpha_1+\alpha_2+\dots+\alpha_n} \cdot \alpha \cdot {x_1}^{\alpha_1} \cdot {x_2}^{\alpha_2} \cdot \dots \cdot {x_n}^{\alpha_n} = t \cdot f(x_1, x_2, \dots, x_n), \forall t > 0$$

so the relation (2) is verified.

One of the most well-known models of the linearly homogeneous functions is that of the Cobb-Douglas production function

$$Q(K, L) = A \cdot K^{\alpha_1} \cdot L^{\alpha_2}, \alpha_1 + \alpha_2 = 1$$

where

-*K* represents capital as a factor of production;
-*L* represent labor as a factor of production;
- α_1, α_2 represents the elasticity in relation to the factors of production.

One of the important issues that attracted attention over time from point of view of economic efficiency was that of obtaining a maximum production volume at a given budget. Mathematically, this problem can be written in the form:

$$\begin{cases} maxU(x_1, x_2, \dots, x_n) \\ \sum_{i=1}^{n} p_i x_i = A \\ x_1, x_2, \dots, x_n \geq 0 \end{cases} \tag{7}$$

where

a) $x_1, x_2, \dots, x_n$ values represent the quantity of goods needed, fixed by the consumer;
b) *U* represents the production function;
c) $p_1, p_2, \dots, p_n$ represent the unit prices of consumer units;
d) *A* represents a fixed budget.

Another issue that attracted attention from an economic pint of view was that of obtaining a given volume of production at a minimum cost. The problem is written in mathematical form as follows:

$$\begin{cases} min(\sum_{i=1}^{n} p_i x_i) \\ Q(x_1, x_2, \dots, x_n) = A = constant \\ x_1, x_2, \dots, x_n \geq 0 \end{cases} \tag{8}$$

where the meaning of the control parameters $p_1, p_2, \dots, p_n$ and the control variables $x_1, x_2, \dots, x_n$ remains the same as in the previous case. But in this case $min(\sum_{i=1}^{n} p_i x_i)$ represents the minimum required expenditure, $Q(x_1, x_2, \dots, x_n)$ represents the production function and $A = constant$ represents the value required for production.

Remark 4 The function that is desired to be minimized in (8) is also a linearly homogeneous function. It is clear that for $f: \mathbb{R}^3 \to \mathbb{R}, f(x_1, x_{2,}, x_3) = p_1 x_1 + p_2 x_2 + p_3 x_3$ we have

$$f(tx_1, tx_2, tx_3) = tf(x_1, x_2, x_3), \forall t > 0$$

As previously specified, we will consider the production function Q as being of the Cobb-Douglas type

$$Q(x_1, x_2, \dots, x_n) = \alpha \cdot x_1^{\alpha_1} \cdot x_1^{\alpha_2} \cdot \dots \cdot x_1^{\alpha_n}, \alpha, \alpha_1, \alpha_2, \dots, \alpha_n > 0$$

in the particular case where

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$$

3. NUMERICAL EXAMPLE

Let us consider the following problem of type (7) but in particular case of three variables:

$$max(120 \cdot {x_1}^{0,3} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2})$$
$$\begin{cases} 90x_1 + 125x_2 + 72x_3 = 1500 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Remark 5 *It is clear if we take into account of (7) that* $x_1, x_{2,}, x_3$ *values represent the quantity of goods needed, fixed by the consumer,* $U(x_1, x_{2,}, x_3) = 120 \cdot {x_1}^{0,3} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2}$ *represents the production function and* $p_1 = 90, p_2 = 125, p_3 = 72$ *represent the unit prices of consumer units and A=1500 represents the fixed budget.*

Solution:

We will solve taking into account relations (4) and (5). In this way we will have

$$f: \mathbb{R}^3 \to \mathbb{R}, f(x_1, x_2, x_3) = 120 \cdot {x_1}^{0,3} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2}$$

and

$$F: \mathbb{R}^3 \to \mathbb{R}, F(x_1, x_2, x_3) = 90x_1 + 125x_2 + 72x_3 - 1500$$

Taking into account (5) we construct the Lagrange function:

$$L: \mathbb{R}^4 \to \mathbb{R}, L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda F(x_1, x_2, x_3) = 120 \cdot {x_1}^{0,3} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2} + \lambda(90x_1 + 125x_2 + 72x_3 - 1500).$$

We determine the stationary points of the function L:

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 0 \Rightarrow 120 \cdot 0{,}3 \cdot {x_1}^{0,3-1} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2} + 90\lambda = 0 \\ \dfrac{\partial L}{\partial x_2} = 0 \Rightarrow 120 \cdot {x_1}^{0,3} \cdot 0{,}5 \cdot {x_2}^{0,5-1} \cdot {x_3}^{0,2} + 125\lambda = 0 \\ \dfrac{\partial L}{\partial x_3} = 0 \Rightarrow 120 \cdot 0{,}2 \cdot {x_1}^{0,3} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2-1} + 72\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow 90x_1 + 125x_2 + 72x_3 - 1500 = 0 \end{cases}$$

which is equivalent with

$$\begin{cases} 120 \cdot 0{,}3 \cdot {x_1}^{0,3-1} \cdot {x_2}^{0,5} \cdot {x_3}^{0,2} = -90\lambda \\ 120 \cdot {x_1}^{0,3} \cdot 0{,}5 \cdot {x_2}^{0,5-1} \cdot {x_3}^{0,2} = -125\lambda \\ 120 \cdot {x_1}^{0,3} \cdot 0{,}5 \cdot {x_2}^{0,5-1} \cdot {x_3}^{0,2} = -72\lambda \\ 90x_1 + 125x_2 + 72x_3 = 1500 \end{cases}$$

Dividing the first equation by the second, then the first equation by the third and performing the calculation we find:

$$5x_2 = 6x_1, 6x_3 = 5x_1$$

and, taking account the relationship

$$90x_1 + 125x_2 + 72x_3 = 1500$$

we get that

$$x_1 = 5, x_2 = 6, x_3 = \frac{25}{6}$$

and the maximum required by the problem is

$$120 \cdot 5^{0,3} \cdot 6^{0,5} \cdot \left(\frac{25}{6}\right)^{0,2} \approx 660{,}14$$

This value represents the maximum production required by the problem given that the budget is set at 1500.

Let us consider the following problem of type (8) but also in particular case of three variables:

$$min(120 \cdot x_1 + 90 \cdot x_2 + 125 \cdot x_3)$$

$$\begin{cases} 64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} = 729 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Remark 6 *It is also clear if we take into account of (8) that* $x_1, x_{2,}, x_3$ *values represent the quantity of goods needed, fixed by the consumer,* $U(x_1, x_{2,}, x_3) = 120\cdot x_1 + 90\cdot x_2 + 125\cdot x_3$ *represents the minimum expenditure required by the problem and* $Q(x_1, x_{2,}, x_3) = 64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3}$ *represent the production function and the value 729 represents the required production volume.*

Solution:

We also will solve taking into account relations (4) and (5). In this way we will have

$$f: \mathbb{R}^3 \to \mathbb{R}, f(x_1, x_2, x_3) = 120\cdot x_1 + 90\cdot x_2 + 125\cdot x_3,$$

respectively

$$F: \mathbb{R}^3 \to \mathbb{R}, F(x_1, x_2, x_3) = 64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} - 72$$

We construct the Lagrange function:

$$L: \mathbb{R}^4 \to \mathbb{R}, L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda\, F(x_1, x_2, x_3) = 120\cdot x_1 + 90\cdot x_2 + 125\cdot x_3 + \lambda(64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} - 729).$$

We determine the stationary points of the function *L*:

$$\begin{cases} \frac{\partial L}{\partial x_1} = 0 \Longrightarrow 120 + 64\cdot \lambda\cdot 0{,}5\cdot {x_1}^{0,5-1}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \Longrightarrow 90 + 64\cdot \lambda\cdot {x_1}^{0,5}\cdot 0{,}2\cdot {x_2}^{0,2-1}\cdot {x_3}^{0,3} = 0 \\ \frac{\partial L}{\partial x_3} = 0 \Longrightarrow 125 + 64\cdot \lambda\cdot {x_1}^{0,5}\cdot {x_2}^{0,2}\cdot 0{,}3\cdot {x_3}^{0,3-1} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \Longrightarrow 64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} - 729 = 0 \end{cases}$$

which is equivalent with

$$\begin{cases} 64\cdot \lambda\cdot 0{,}5\cdot {x_1}^{0,5-1}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} = -120 \\ 64\cdot \lambda\cdot 0{,}2\cdot {x_1}^{0,5}\cdot {x_2}^{0,2-1}\cdot {x_3}^{0,3} + 90\lambda = -90 \\ 64\cdot \lambda\cdot 0{,}3\cdot {x_1}^{0,5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3-1} + 90\lambda = -125 \\ 64\cdot {x_1}^{0.5}\cdot {x_2}^{0,2}\cdot {x_3}^{0,3} = 720 \end{cases} \tag{10}$$

Dividing the first equation by the second, then the first equation by the third and performing the calculation we find:

$$15x_2 = 8x_1, 125x_3 = 72x_1$$

and, taking account the relationship

$$64 \cdot {x_1}^{0.5} \cdot {x_2}^{0,2} \cdot {x_3}^{0,3} = 729$$

we get that:

$$x_1 = \frac{729}{64 \cdot \left(\frac{8}{15}\right)^{0,2} \cdot \left(\frac{72}{125}\right)^{0,3}} \approx 15{,}25, x_2 \approx 8{,}12, x_3 \approx 8{,}24$$

and the minimum required by the problem will be

$$125 \cdot 15{,}24 + 90 \cdot 8{,}12 + 125 \cdot 8{,}77 = 3655{,}85$$

This value represents the minimum cost required by the problem that can be obtained when the required production volume of 729 is required to be reached in the case of the present example.

Remark 7 I*n the systems described by relations (9) and (10), when the first and second equations, respectively the first and third, were subsequently divided, the resulting values of the calculations were written in the final version without further insisting on the concrete way of performing the calculations.*

4. CONCLUSIONS

The present paper focuses on linearly homogeneous models having particular Cobb-Douglas-type functions in which the sum of the parameters is equal to 1.

Two economic models in which Cobb-Douglas production functions appear are proposed for study in the present paper after the preliminary presentation of the classical Cobb-Douglas production function having as variables the production factors labor and capital.

The first of these, related to obtaining a maximum production volume at a given budget, is exemplified with a Cobb-Douglas production function and subsequently a numerical example with such a production function in three variables is presented.

A second such model in which the production function is linear, implicitly a linearly homogeneous one, related to obtaining a given volume of production at a minimum cost, is presented below. A numerical example in three variables is also subsequently proposed and solved.

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